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STABILITY ANALYSIS IN A LEONTIEF OPTIMAL GROWTH MODEL

by

CHIA-SHIN CHUNG

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STABILITY ANALYSIS IN A LEONTIEF OPTIMAL GROWTH MODEL,

by

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DEDICATION

I dedicate this dissertation to my fiancée Su-Lin, whose gracefulness I shall cherish forever.

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#### ABSTRACT

It is shown in our model that there are two types of optimal stationary programs--boundary type and interior type. There exists a critical discount factor  $\delta'$  for each type of optimal stationary program such that when the discount factor  $\delta$  is less than  $\delta'$ , the corresponding optimal program will be unstable. Examples are given to demonstrate this.

## TABLE OF CONTENTS

	Page
CHAPTER 1: INTRODUCTION . . . . .	1
CHAPTER 2: THE MODEL AND OPTIMAL STATIONARY PROGRAM . . . . .	3
2.1 The Model . . . . .	3
2.2 Optimal Stationary Program . . . . .	9
2.3 Existence of Competitive Prices . . . . .	18
CHAPTER 3: STABILITY ANALYSIS OF AN OPTIMAL STATIONARY PROGRAM .	23
3.1 Introduction . . . . .	23
3.2 Instability Property of Boundary OSP . . . . .	24
3.3 Instability Property of Interior OSP . . . . .	30
3.4 An Example . . . . .	37
3.5 General 2-sector Model . . . . .	38
REFERENCES . . . . .	40

CHAPTER 1  
INTRODUCTION

We started to work on this paper with the interest of studying the sensitivity analysis of a convex programming problem over infinite horizon with discounted utility. It turns out that in our simple model we are able to find some conditions such that if these conditions hold, the optimal stationary programs will be unstable. Sutherland [7] gave a similar example in his thesis, but it is not easy to interpret.

There have been discussions of how the optimal program can be a continuous function of the initial stock that it starts with, under a suitable topology defined on the space of all feasible programs. Takekuma [8] proved it in the case of weak\* topology, but that is not sufficient to show the stability property of the steady state. We need a finer topology like the one induced by  $L_\infty$  norm. Araujo and Scheinkman [1] showed that this is indeed possible under the condition that the steady state lies in the interior of the technology and the utility function satisfies some dominant diagonal property, which is needed for applying the implicit function theorem in the proof. Yet we know very little about the real meaning of these conditions. Evers [2], in a model similar to ours but with linear utility function, proved that every optimal stationary program is stable by showing that the optimal program is a continuous function of the initial condition in the sense of  $L_\infty$  norm, but he made an serious error in his proof. Our result could be served as an counterexample to his theorem. Nevertheless we are benefited from some of his ideas.

In a different approach, Scheinkman [6] proved that for the

discount factor  $\delta$  close enough to 1, stability property holds, but we still don't know the behavior of the optimal program for  $\delta$  away from 1. The examples we present in our paper show that when  $\delta$  is less than some critical discount factor  $\delta'$ , which is less than 1, the optimal stationary program will be unstable. We have shown this for each type of optimal stationary programs as will be defined in the next chapter. For more references concerning this problem, see McKenzie [5].

Also included in this paper are some interesting results about the total number of optimal stationary programs in our model. We show that the total number has to be odd; and under a special condition, it is in fact unique. Because we try to make this paper as self-contained as possible, we have confined our result to a 2-sector model, even though some theorems in our paper can actually be generalized to n-sector model without much effort. We will try to expand some of them in the future when it is worthwhile to do so.

In Chapter 2, we give a description of the model and study the behavior of the optimal stationary program. We have shown that there are two types of optimal stationary programs--boundary type and interior type, we have also shown that the existence of competitive prices for any optimal program can be obtained from Weitzman [8]. Chapter 3 shows that instability theorems for both types of optimal stationary programs could be obtained by designing a two-stage argument--forward argument and backward argument.

CHAPTER 2  
THE MODEL AND OPTIMAL STATIONARY PROGRAM

2.1 The Model

In our model, there are two production goods and one resource which is labor. The technology consists of two activities, each of which produces exactly one production good but could use various amounts of production goods and labor. It is natural to assume that the  $i$ th activity produces the  $i$ th production good. In each period, the production goods produced in the previous period are used in two ways. Some for pure consumption and the rest for production of production goods for the next period. Utility function is assumed to be a concave  $C^2$ -differentiable function of the production goods consumed in each period and is assumed to be the same for every period. We also assume a constant supply of labor in each period, which is one, and also a constant amount of labor used to operate activity  $j$  at unit level, which is also one. The objective of this model is to choose a feasible program which maximizes the discounted sum of future utility.

To formulate the model mathematically, we will need the following notations.

Let  $\delta \in (0,1)$  be the utility discount factor and

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{R}_+^{2 \times 2}$$

where  $a_{ij}$  denotes the amount of good  $i$  used to operate activity  $j$  at unit level.

Utility function  $u(c_1, c_2)$  defined on  $\mathbb{R}_+^2$  is assumed to be strictly concave and  $C^2$ -differentiable and, furthermore,  $u_1$  and  $u_2$  are assumed to be positive, here  $u_i \equiv \frac{\partial u}{\partial c_i}$   $i = 1, 2$ .

The model can now be given as follows:

Given an initial stocks  $X(0)$ , find a program  $(X(t), C(t))_{t=0}^{\infty}$  which solve the following optimization problem.

$$\max \sum_{t=0}^{\infty} \delta^t u(C(t))$$

Subject to:

$$(2.1) \quad AX(t+1) + C(t) \leq X(t)$$

$$(2.2) \quad eX(t+1) \leq 1$$

$$X(t) \geq 0 \quad t = 1, 2, \dots$$

$$C(t) \geq 0 \quad t = 0, 1, 2, \dots$$

Note:  $e = (1, 1)$

We will need the following definitions.

Definition:

A program  $(X(t), C(t))_{t=0}^{\infty}$  starting from  $X(0)$  is feasible if it satisfies the above constraints.

Definition:

A feasible program is *optimal* if it maximizes  $\sum_{t=1}^{\infty} \delta^t u(C(t))$  over the set of all feasible programs starting from the same initial stocks.

Definition:

A feasible program is an *optimal stationary program* (OSP), if it is optimal and  $(X(t), C(t)) = (X, C)$  for  $t \geq 0$  for some constant  $(X, C)$ .

Definition:

A feasible program  $(X(t), C(t))_{t=0}^{\infty}$  is *competitive* if there exists a set of nonnegative prices  $(P(t), W(t))_{t=0}^{\infty}$  where  $P(t) = (P_1(t), P_2(t)) \in \mathbb{R}_+^2$  and  $W(t) \in \mathbb{R}_+$ , which satisfies the following constraints.

$$(2.3) \quad P(t)A + W(t)e \geq P(t+1)$$

$$(2.4) \quad P(t) \geq \delta^t \nabla u(C(t))$$

$$(2.5) \quad C(t)(P(t) - \delta^t \nabla u(C(t))) = 0$$

$$(2.6) \quad X(t+1)(P(t)A + W(t)e - P(t+1)) = 0$$

$$(2.7) \quad P(t)(X(t) - AX(t+1) - C(t)) = 0$$

$$(2.8) \quad (1 - eX(t+1)) = 0$$

$$(2.9) \quad P(t)AX(t+1) \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

We begin with the following existence lemma.

Lemma 2.1

Given any stocks  $X(0)$ , there exists an optimal program starting from  $X(0)$ .

Proof:

Let  $(X(t), C(t))_{t=0}^{\infty}$  be any feasible program starting from  $X(0)$ .

(2.1) implies  $C(t) \leq X(t)$ , this together with (2.2) shows that  $(X(t), C(t))_{t=0}^{\infty}$  is a bounded sequence. Since  $u$  is  $C^2$ -differentiable, there exists  $M > 0$  (2.10) such that

$$(2.10) \quad |u(C(t))| \leq M \text{ for all } t.$$

We define

$$U(X(0)) = \sup \left\{ \sum_{t=0}^{\infty} \delta^t u(C(t)) \mid (X(t), C(t))_{t=0}^{\infty} \text{ is any feasible program starting from } X(0) \right\}$$

By (2.10), we have

$$\left| \sum_{t=0}^{\infty} \delta^t u(C(t)) \right| \leq \sum_{t=0}^{\infty} \delta^t M = \frac{M}{1 - \delta}$$

hence

$$|U(X(0))| \leq \frac{M}{1 - \delta}.$$

Convergence of  $\sum_{t=0}^{\infty} \delta^t M$  implies, for any  $\epsilon_1 > 0$ , there exists  $T_1 > 0$  such that

$$(2.11) \quad \sum_{t=t_0}^{\infty} \delta^t M < \epsilon_1 \text{ for } t_0 > T_1.$$

By definition of  $U(X(0))$ , there exists a sequence of feasible programs  $(X_n(t), C_n(t))_{t=0}^{\infty}$   $n = 0, 1, \dots$  starting from  $X(0)$ , such that  $\sum_{t=0}^{\infty} \delta^t u(C_n(t))$  converges to  $U(X(0))$  as  $n \rightarrow \infty$  or for any  $\epsilon_2 > 0$ , there exists  $N_2 > 0$  such that

$$(2.12) \quad \left| \sum_{t=0}^{\infty} \delta^t u(C(t)) - U(X(0)) \right| < \epsilon_2 \quad \text{for } n > N_2 .$$

We know  $\{(X_n(t), C_n(t))\}_{n=1}^{\infty}$  is a bounded sequence for each  $t$ . By diagonal converging process, we can find a converging subsequence converging to a limiting program,  $(X_{\infty}(t), C_{\infty}(t))_{t=0}^{\infty}$ . It can be easily shown that  $(X_{\infty}(t), C_{\infty}(t))_{t=0}^{\infty}$  is a feasible program. If we can show that  $\sum_{t=0}^{\infty} \delta^t u(C_{\infty}(t)) = U(X(0))$ , then  $(X_{\infty}(t), C_{\infty}(t))_{t=0}^{\infty}$  is optimal and we are done. By the property of diagonal converging process, given any  $T > 0$  and  $\epsilon_3 > 0$ , there exists  $N_3 > 0$  such that

$$(2.13) \quad \left| \sum_{t=0}^T \delta^t u(C_n(t)) - \sum_{t=0}^T \delta^t u(C_{\infty}(t)) \right| < \epsilon_3 \quad \text{for } n > N_3$$

hence if we choose  $T > T_1$ , then by (2.11), (2.12) and (2.13), for  $n$  large enough we have

$$\begin{aligned} & \left| \sum_{t=0}^{\infty} \delta^t u(C_{\infty}(t)) - U(X(0)) \right| \\ & \leq \left| \sum_{t=0}^{\infty} \delta^t u(C_{\infty}(t)) - \sum_{t=0}^{\infty} \delta^t u(C_n(t)) \right| + \left| \sum_{t=0}^{\infty} \delta^t u(C_n(t)) - U(X(0)) \right| \\ & \leq \left| \sum_{t=0}^T \delta^t u(C_{\infty}(t)) - \sum_{t=0}^T \delta^t u(C_n(t)) \right| + \left| \sum_{t=T}^{\infty} \delta^t u(C_{\infty}(t)) \right| \\ & \quad + \left| \sum_{t=T}^{\infty} \delta^t u(C_n(t)) \right| + \epsilon_2 \\ & \leq 2\epsilon_1 + \epsilon_2 + \epsilon_3 . \end{aligned}$$

Since  $\epsilon_1, \epsilon_2, \epsilon_3$  can be made arbitrary small, this concludes our proof. ■

The following lemma gives a sufficient condition for an optimal program.

Lemma 2.2

Any competitive program is optimal.

Proof:

Let  $(\bar{X}(t), \bar{C}(t))_{t=0}^{\infty}$  be any competitive program. By definition, there exists a set of nonnegative prices  $(P(t), W(t))_{t=0}^{\infty}$  satisfying (2.3) - (2.9). Let  $(X(t), C(t))_{t=0}^{\infty}$  be any feasible program starting from the same initial stock. Given any  $T > 0$ , we can derive the following

$$\begin{aligned}
 & \sum_{t=0}^T \delta^t u(\bar{C}(t)) - \sum_{t=0}^T \delta^t u(C(t)) \\
 & \geq \sum_{t=0}^T \delta^t \nabla u(\bar{C}(t))(\bar{C}(t) - C(t)) \quad (\text{by concavity of } u) \\
 & \geq \sum_{t=0}^T P(t)(\bar{C}(t) - C(t)) \quad (\text{by (2.4) and (2.5)}) \\
 & \geq \sum_{t=0}^T P(t)(\bar{X}(t) - A\bar{X}(t+1) - X(t) + AX(t+1)) \quad (\text{by (2.1) and (2.7)}) \\
 & = P(0)(\bar{X}(0) - X(0)) \\
 & + \sum_{t=0}^{T-1} (P(t+1) - P(t)A)(\bar{X}(t+1) - X(t+1)) \\
 & + P(T)A(X(T+1) - \bar{X}(T+1)) \quad (\text{rearrangement of terms}) \\
 & \geq \sum_{t=0}^{T-1} W(t)e(\bar{X}(t+1) - X(t+1)) \\
 & + P(T)A(X(T+1) - \bar{X}(T+1)) \quad (\text{by (2.3) and (2.6)})
 \end{aligned}$$

$$\begin{aligned}
 &\geq \sum_{t=0}^{T-1} W(t)(1 - eX(t+1)) \\
 &\quad + P(T)A(X(T+1) - \bar{X}(T+1)) \quad (\text{by (2.8)}) \\
 &\geq P(T)A(X(T+1) - \bar{X}(T+1)) \quad (\text{by (2.2)}) \\
 &\geq -P(T)A\bar{X}(T+1) .
 \end{aligned}$$

Now let  $T$  tend to  $\infty$ , (2.2) and (2.9) imply that  $-P(T)A\bar{X}(T+1)$  would go to 0. Hence

$$\sum_{t=0}^{\infty} \delta^t u(\bar{C}(t)) - \sum_{t=0}^{\infty} \delta^t u(C(t)) \geq 0 .$$

which means  $(\bar{X}(t), \bar{C}(t))_{t=0}^{\infty}$  is optimal. ■

## 2.2 Optimal Stationary Program

In this section, we will study the optimal stationary programs in our model. We need to restrict our model to be  $\delta$ -productive, whose definition is given as follows.

### Definition:

A is said to be  $\delta$ -productive, if there exists  $X \geq 0$  such that  $(\delta I - A)X > 0$ . In the case  $\delta = 1$ , we say it is productive.

It is well known that A is  $\delta$ -productive if and only if  $\delta I - A$  is Leontief.

Characteristically, we will show that there are only two types of OSPs, they are defined below.

### Definition:

An OSP  $(X, C)$  is called a boundary OSP if either  $c_1 = 0$  or  $c_2 = 0$ .

It is called an interior OSP if both  $c_1$  and  $c_2$  are positive.

To explore these two types of OSPs, we need to establish a necessary and sufficient conditions for an OSP.

Lemma 2.3

$(X, C)$  is an OSP if and only if there exists  $(p_\delta, w_\delta) \geq 0$  such that the following conditions are satisfied

$$(2.14) \quad (a) \quad \begin{pmatrix} A - I & e \\ e & 0 \end{pmatrix} \begin{pmatrix} X \\ C \end{pmatrix} \leq \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (\text{primal})$$

$$X \geq 0 \quad C \geq 0$$

$$(2.15) \quad (b) \quad (p_\delta, w_\delta) \begin{pmatrix} A - \delta I & e \\ e & 0 \end{pmatrix} \geq \begin{pmatrix} 0 \\ \nabla u(C) \end{pmatrix} \quad (\text{dual})$$

$$(2.16) \quad (c) \quad (p_\delta, w_\delta) \left( \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} A - I & e \\ e & 0 \end{pmatrix} \begin{pmatrix} X \\ C \end{pmatrix} \right) = 0 \quad (\text{complementarity})$$

$$\left| (p_\delta, w_\delta) \begin{pmatrix} A - \delta I & e \\ e & 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \nabla u(C) \end{pmatrix} \right| \begin{pmatrix} X \\ C \end{pmatrix} = 0$$

Theorem 1 in Hansen and Koopmans [3] gave a necessary and sufficient condition for any OSP which is more than reproducible (see the definition on page 494 in [3]). But Lemma 6.1 in Jones [4] shows that any OSP in our model is more than reproducible. Lemma 2.3 can then be obtained by rewriting Theorem 1 in [3] in our terminology. We will not give a formal proof here.

Note: (1) If  $(X, C, p_\delta, w_\delta)$  satisfies (a), (b) and (c) in Lemma 2.3, then the stationary program  $(X, C)$  is competitive and is supported by  $(P(t), W(t)) \equiv \delta^t (p_\delta, w_\delta)$ .

(2) By assumption on  $u$  and Lemma 2.3(b), we have  $p_\delta \geq \nabla u(C) > 0$ . hence (c) implies

(2.17)

$$(A - I)X + C = 0 .$$

Lemma 2.4

Let  $(X, C)$  be an OSP, then  $x_1 + x_2 = 1$ , here  $X = (x_1, x_2)$ .

Proof:

If  $x_1 + x_2 < 1$ , then by (2.16),  $w_\delta = 0$ . (3.15) implies  $p_\delta(\delta I - A) \leq 0$  and  $p_\delta \geq \nabla u(C) > 0$ . By assumption,  $\delta I - A$  is a Leontief matrix, this implies that  $(\delta I - A)^{-1} \geq 0$  hence  $p_\delta = p_\delta(\delta I - A)(\delta I - A)^{-1} \leq 0$ , which is a contradiction. ■

We need to make one more assumption.

Assumption:

$A$  is indecomposable, i.e.,  $a_{12}$  and  $a_{21}$  have to be positive.

From Lemma 2.4 we can see that for the model to have boundary OSPs, they should come from the solutions of the following two systems of equations

$$(I) \quad \left\{ \begin{array}{l} (A - I)X + C = 0 \\ x_1 + x_2 = 1 \\ c_1 = 0 \end{array} \right.$$

$$(II) \quad \left\{ \begin{array}{l} (A - I)X + C = 0 \\ x_1 + x_2 = 1 \\ c_2 = 0 . \end{array} \right.$$

We can compute their respective solutions as follows

$$(x^1, c^1) = \left( \frac{1 - a_{22}}{1 + a_{21} - a_{22}}, \frac{a_{21}}{1 + a_{21} - a_{22}}, \frac{\det(A - I)}{1 + a_{21} - a_{22}}, 0 \right)$$

$$(x^2, c^2) = \left( \frac{a_{12}}{1 + a_{12} - a_{11}}, \frac{1 - a_{11}}{1 + a_{12} - a_{11}}, 0, \frac{\det(A - I)}{1 + a_{12} - a_{11}} \right).$$

The following theorem tells when  $(x^1, c^1)$  or  $(x^2, c^2)$  will be an OSP.

Theorem 2.1

$$(a) (x^1, c^1) \text{ is an OSP iff } \frac{u_1(c^1)}{u_2(0)} \geq \frac{\delta + a_{21} - a_{22}}{\delta + a_{12} - a_{11}}$$

$$(b) (x^2, c^2) \text{ is an OSP iff } \frac{u_1(0)}{u_2(c^2)} \leq \frac{\delta + a_{21} - a_{22}}{\delta + a_{12} - a_{11}}.$$

Proof:

Since  $A$  is assumed to be indecomposable and productive, it implies  $x^1 > 0$  and  $c^1 > 0$ . Therefore, by the complementarity conditions in (2.16),  $(p_\delta^1, w_\delta^1)$  can be uniquely solved and we have

$$(p_\delta^1, w_\delta^1) = \left( u_1(c^1), \frac{\delta + a_{12} - a_{11}}{\delta + a_{21} - a_{22}} u_1(c^1), \frac{\det(\delta I - A)}{\delta + a_{21} - a_{22}} u_1(c^1) \right).$$

For  $(p_\delta^1, w_\delta^1)$  to satisfy (2.15) and (2.16), it is necessary and sufficient that

$$\frac{\delta + a_{12} - a_{11}}{\delta + a_{21} - a_{22}} u_1(c^1) \geq u_2(0)$$

or

$$\frac{u_1(C^1)}{u_2(0)} \geq \frac{\delta + a_{21} - a_{22}}{\delta + a_{12} - a_{11}}$$

this proves (a). (b) can be shown similarly. ■

From the proof of the above theorem, we also observe that the nonnegative vectors  $(p_\delta, q_\delta)$  obtained for (a) and (b) are uniquely determined. We will denote them as  $(\bar{P}^1, \bar{W}^1)$  and  $(\bar{P}^2, \bar{W}^2)$  for  $(X^1, C^1)$  and  $(X^2, C^2)$  respectively.

To find interior OSPs, it turns out that they have to be convex combination of the boundary OSPs obtained above.

### Theorem 2.2

A stationary program  $(X, C)$  is an interior OSP iff there exists  $\lambda \in (0,1)$  such that

$$(a) \quad (X, C) = (1 - \lambda)(X^1, C^1) + \lambda(X^2, C^2)$$

$$(b) \quad \frac{u_1(C)}{u_2(C)} = \frac{\delta + a_{21} - a_{22}}{\delta + a_{12} - a_{11}}.$$

#### Proof:

"sufficiency"

It can be shown that (a) and (b) imply  $(X, C)$  is feasible and is supported by  $\left( u_1(C), u_2(C), \frac{\det(\delta I - A)}{\delta + a_{21} - a_{22}} u_1(C) \right)$  in the sense of

Lemma 2.3, hence  $(X, C)$  is optimal. Since  $C$  is positive,  $(X, C)$  is an interior OSP.

"necessity"

Let  $(X, C)$  be any interior OSP. By Lemma 2.4,  $x_1 + x_2 = 1$ , hence

we can eliminate  $x_1, x_2$  from (2.17) and write  $c_1$  as a decreasing function of  $c_2$ . Since  $c_1 < c_1^1$ , it implies there exists  $\lambda \in (0,1)$  such that  $c_1 = (1 - \lambda)c_1^1$ . Using (2.17) again implies  $(X, C) = (1 - \lambda)(X^1, C^1) + \lambda(X^2, C^2)$  is the unique stationary program having that property, this proves (a). (b) follows immediately from (2.16). ■

Combining Theorems 2.1 and 2.2, we will now be able to count the total number of OSPs.

Theorem 2.3

Except in degenerate cases, the total number of OSPs is odd.

Proof:

To compute the total number of OSPs, Theorems 2.1 and 2.2 suggest that we look at  $\frac{u_1}{u_2}$  as a function of  $\lambda$ . We already know that there are only two boundary OSPs. We now look at each possible case that could happen.

Case 1 Only one of  $(X^1, C^1)$  and  $(X^2, C^2)$  is an OSP.

By Theorem 2.1, both endpoints of the graph of  $\frac{u_1}{u_2}$  are either above or below the horizontal line  $y = \frac{\delta + a_{21} - a_{22}}{\delta + a_{12} - a_{11}}$  simultaneously.

Hence, except in degenerate case, the graph of  $\frac{u_1}{u_2}$  can cross that line

only even number of times. Which by Theorem 2.2, means even number of interior OSPs; therefore, the total number of OSPs is odd. See Figure 1. ("o" in the graph shows where the OSPs are located.)

Case 2 Both  $(X^1, C^1)$  and  $(X^2, C^2)$  are either OSPs or both are not.

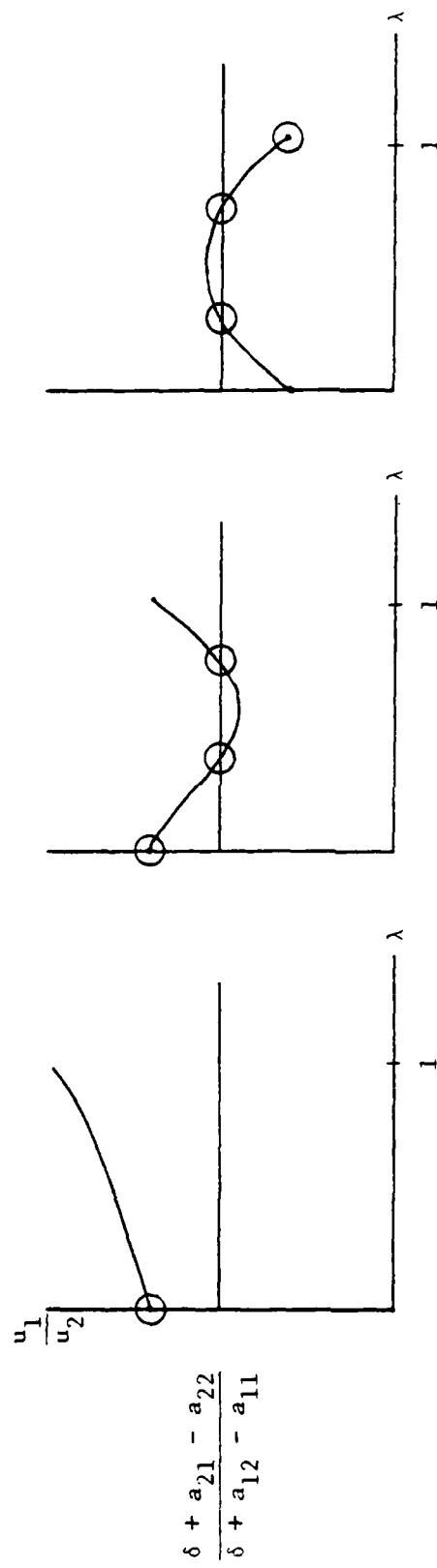


FIGURE 1(a)

FIGURE 1(b)

FIGURE 1(c)

This implies one and only one endpoint of the graph of  $\frac{u_1}{u_2}$  is above the line  $y = \frac{\delta + a_{21} - a_{22}}{\delta + a_{12} - a_{11}}$ . Hence, except in degenerate case, the graph of  $\frac{u_1}{u_2}$  could cross that line only odd number of times. Hence, the total number of OSPs is again odd. See Figure 2. ■

We also find the following uniqueness theorem.

Theorem 2.4

Assuming that both production goods are complementary i.e.,  $u_{12}(C) \geq 0$  for all  $C \geq 0$ , then there exists an unique OSP--either a boundary OSP or an interior OSP.

Proof:

Let  $C = (1 - \lambda)c^1 + \lambda c^2$  define

$$f(\lambda) \equiv \frac{u_1(C)}{u_2(C)} = \frac{u_1((1 - \lambda)c^1 + \lambda c^2)}{u_2((1 - \lambda)c^1 + \lambda c^2)}.$$

$f'(\lambda)$  would then be

$$\frac{1}{(u_2(C))^2} \left[ u_2(C)(-u_{11}(C)c^1_1 + u_{12}(C)c^2_1) - u_1(C)(-u_{21}(C)c^1_1 + u_{22}(C)c^2_1) \right],$$

hence,  $f'(\lambda) > 0$  for  $\lambda \in (0,1)$ . Therefore, the graph of  $\frac{u_1}{u_2}$  is increasing. This implies it either crosses the line  $y = \frac{\delta + a_{21} - a_{22}}{\delta + a_{12} - a_{11}}$  exactly once or lying entirely above or below that line. In each case, there is only one OSP. ■

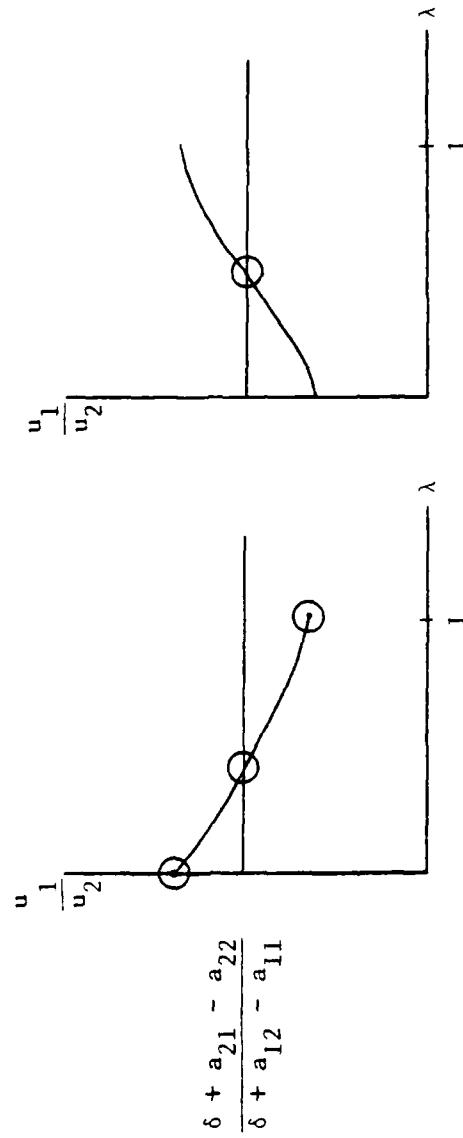


FIGURE 2(a)

FIGURE 2(b)

Corollary:

If  $u$  is separable, then there exists an unique OSP.

In Chapter 3, we will assume that utility is separable. Hence we will be dealing only with a single OSP--either a boundary OSP or an interior OSP.

The following example shows that if the condition in Theorem 2.4 is violated, then it is possible to have more than one OSP.

Example:

Let  $u(c_1, c_2) = \log(1.3c_1 + c_2) + 0.01c_1$  and  $A = \begin{pmatrix} 0 & 1 \\ .8 & 0 \end{pmatrix}$ , then

$$(x^1, c^1) = \left( \frac{5}{9}, \frac{4}{9}, \frac{3.4}{9}, 0 \right)$$

$$(x^2, c^2) = \left( \frac{2}{7}, \frac{5}{7}, 0, \frac{3.4}{7} \right).$$

$u(c_1, c_2)$  can be shown to be strictly concave, but

$$u_{12} = \frac{-1.3}{(1.3c_1 + c_2)^2} < 0.$$

By Theorem 2.1, if  $\delta$  satisfies  $1.3 + 0.013c_1^1 > \frac{\delta + .8}{\delta + .4} > 1.3 + .01c_2^2$ , then both boundary stationary programs will be optimal. It can be easily checked that the above constraint for  $\delta$  is feasible.

2.3 Existence of Competitive Prices

In this section, we will show that there exists a set of competitive prices for any optimal program. Weitzman [9] established a duality theorem in an infinite horizon convex programming to characterize an optimal program. It turns out that his result can be adapted to our model.

First, we introduce the following notations

$$\tilde{A} = \begin{pmatrix} A & I \\ e & 0 \end{pmatrix} \in \mathbb{R}_+^{3 \times 4} \quad \tilde{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}_+^3$$

$$\tilde{B} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}_+^{3 \times 4} \quad \tilde{x} = \begin{pmatrix} x_1 \\ x_2 \\ c \end{pmatrix} \in \mathbb{R}_+^3$$

An induced utility function  $\tilde{u}$  of  $u$  is defined for all  $\tilde{x} \in \mathbb{R}_+^3$  such that  $\tilde{u}(\tilde{x}) = u(c)$ . To fit our model into that of Weitzman's, we define a technology  $Q$  in each period as

$$Q_t = \{(g, u, h) \mid g \geq \tilde{A}\tilde{x} - \tilde{b}, u \leq \delta^{t-1}\tilde{u}(\tilde{x}) \text{ and } h \leq \tilde{B}\tilde{x} \\ \text{for } \tilde{x} \geq 0, g \geq 0 \text{ and } h \geq 0\}.$$

It is easy to see that  $Q$  satisfies conditions  $^o_1 - ^o_4$  in [9], but not the reachability condition, i.e.,  $^o_5$ . Careful reading of Weitzman's proof reveals that condition  $^o_5$  is needed only to show that for  $T > 0$   $\psi_T(X)$  is finite for all  $X \geq 0$ . Where, in our terminology,  $\psi_T(X)$  is defined to be  $\sup_{t=T+1}^{\infty} \delta^t u(C(t))$  over the set of all feasible programs  $(x(t), C(t))_{t=T+1}^{\infty}$  starting from some initial stock. But the proof of Lemma 2.1 has shown that  $\sum_{t=T+1}^{\infty} \delta^t u(C(t))$  is always bounded, hence  $\psi_T(X)$  is finite. Hence  $^o_5$  is not needed in our model. Weitzman's theorem can be restated as follows:

Let  $(u_t, h(t))_{t=1}^{\infty}$  be an optimal program with initial stock  $g_0$ , then there exists nonnegative prices  $(P(t))_{t=1}^{\infty}$  such that

$$(2.18) \quad u_1 + \tilde{P}(1)h(1) \geq u + \tilde{P}(1)h \quad (g_0, u, h) \in Q$$

$$(2.19) \quad u_t + \tilde{P}(t)h(t) - \tilde{P}(t-1)h(t-1) \geq u + \tilde{P}(t)h - \tilde{P}(t-1)g \\ (g, u, h) \in Q \text{ for } t \geq 2$$

$$(2.20) \quad \lim_{t \rightarrow \infty} \tilde{P}(t)h(t) = 0.$$

Let  $\tilde{P}(t) = (P(t), W(t)) \in \mathbb{R}_+^3$  where  $P(t) \in \mathbb{R}_+^2$  and  $W(t) \in \mathbb{R}_+$ .

### Theorem 2.5

Any optimal program is competitive.

#### Proof:

Let  $(X(t), C(t))_{t=0}^{\infty}$  be any optimal program with initial stock  $X(0)$  for our model, then the program  $(\delta^t u(C(t)), \tilde{B}X(t))_{t=0}^{\infty}$  with initial stock  $g_0 = \begin{pmatrix} X(0) \\ 0 \end{pmatrix}$  is optimal for  $Q$ , where  $\tilde{X}(t)$  is defined to be  $(X(t), C(t-1))$ . Hence there exists a set of nonnegative prices  $(\tilde{P}(t))_{t=1}^{\infty}$  such that (2.18), (2.19) and (2.20) hold. We will show that  $(\tilde{P}(t))_{t=1}^{\infty}$  is the set of competitive prices that we are looking for. We will show it in two cases.

#### Case 1 $t = 1$

(2.18) can be restated as follows

$$u(C(0)) + \tilde{P}(1)\tilde{B}\tilde{X}(1) \geq \tilde{u}(\tilde{X}) + \tilde{P}(1)\tilde{B}\tilde{X} \text{ for all } \tilde{X} \geq 0$$

such that  $g_0 \geq \tilde{A}\tilde{X} - \tilde{b}$ . By Kuhn-Tucker Theorem, there exists  $\tilde{P}(0) \geq 0$  such that (2.3) - (2.9) hold for  $t = 0$ .

Case 2  $t \geq 2$

(2.19) implies

$$(2.21) \quad \begin{aligned} & \delta^{t-1} u(C(t-1)) + \tilde{P}(t) \tilde{B}\tilde{X}(t) - \tilde{P}(t-1) \tilde{B}\tilde{X}(t-1) \\ & \geq u_t + \tilde{P}(t) \tilde{B}\tilde{X} - \tilde{P}(t-1)(\tilde{A}\tilde{X} - \tilde{b}) \quad \text{for all } \tilde{X} \geq 0. \end{aligned}$$

Now  $\tilde{B}\tilde{X}(t-1) = \tilde{A}\tilde{X}(t) - \tilde{b} + Z(t-1)$  for some  $Z(t-1) \geq 0$ , (2.21) becomes

$$(2.22) \quad \begin{aligned} & \delta^{t-1} u(C(t-1)) + \tilde{P}(t) \tilde{B}\tilde{X}(t) - \tilde{P}(t-1) \tilde{A}\tilde{X}(t) - \tilde{P}(t-1) Z(t-1) \\ & \geq u_t + \tilde{P}(t) \tilde{B}\tilde{X} - \tilde{P}(t-1) \tilde{A}\tilde{X}. \end{aligned}$$

Plug in  $(u_t, \tilde{X}) = (\delta^{t-1} u(C(t-1)), \tilde{X}(t))$  we can show  $\tilde{P}(t-1) Z(t-1) = 0$ , which is just (2.7) and (2.8) for  $t \geq 1$ . The first order maximum conditions for (2.22) then gives (2.3) - (2.6) for  $t \geq 1$ . (2.9) follows easily from (2.20). ■

Definition:

$(X(t), C(t))_{t=0}^{\infty}$  is called a stable optimal program if it is optimal and  $(X(t), C(t))$  converges to an OSP as  $t$  tends to  $\infty$ .

For the purpose of stability analysis in the next chapter, competitive prices for a stable optimal program need to possess the following property.

Theorem 2.6

Let  $(P(t), W(t))_{t=0}^{\infty}$  be any competitive prices for a stable optimal program  $(X(t), C(t))_{t=0}^{\infty}$ , then

$$\{\delta^{-t} (P(t), W(t))\}_{t=0}^{\infty}$$

is a bounded sequence.

Proof:

Since  $A$  is assumed to be indecomposable,  $(X(t), C(t))_{t=0}^{\infty}$  can never be stable if  $x_i(t) = 0$  for some  $i$  and  $t$ , hence (2.6) implies

$$(2.23) \quad P(t)A + W(t)e - P(T+1) = 0 \quad \text{for } t \geq 0.$$

Without loss of generality, we assume that  $(X(t), C(t)) \rightarrow (X^1, C^1)$  as  $t \rightarrow \infty$ , hence there exists  $T > 0$  such that  $c_1(t) > 0$  for  $t > T$ . (2.5) then implies  $p_1(t) = \delta^t u_1(C(t))$  for  $t > T$ , hence

$\{\delta^{-t} p_1(t)\}_{t=T+1}^{\infty}$  is a bounded sequence. (2.18) and indecomposability of  $A$  imply that  $\{\delta^{-t} (p(t), W(t))\}_{t=0}^{\infty}$  is a bounded sequence. ■

## CHAPTER 3

## STABILITY ANALYSIS OF AN OPTIMAL STATIONARY PROGRAM

3.1 Introduction

In this chapter, we study the instability property of a special two-sector model. We choose this special model for it's simplicity. We will also discuss how it can be extended to any two-sector model at the end.

We will need separate arguments of instability property for each type of OSP. For each type, we will show that there exists a critical discount factor  $\delta'$  such that when  $\delta < \delta'$ , the OSP is unstable. In the case of boundary OSP,  $\delta'$  depends only on  $A$ , while for the interior OSP it also depends on the utility function. We assume that the utility is separable, hence by the corollary in Chapter 2, there exists an unique OSP.

The instability argument is composed of two parts. First part shows that for  $\delta < \delta'$ , any stable optimal program would have to lie on the turnpike eventually, we call it the forward argument. Second part shows that for an optimal program to have the above property, it can only start from a set of measure 0, this is called the backward argument.

Example will be given toward the end of this Chapter.

We now describe our special two-sector model and rewrite the constraints and competitive conditions again. Let  $A = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$  and  $u(c_1, c_2) = \phi_1(c_1) + \phi_2(c_2)$ .

Primal constraints are

$$(3.1) \quad \alpha x_2(t+1) + c_1(t) = x_1(t)$$

$$(3.2) \quad \beta x_1(t+1) + c_2(t) = x_2(t)$$

$$(3.3) \quad x_1(t+1) + x_2(t+1) \leq 1$$

$$x(t), c(t) \geq 0 \text{ for } t \geq 0$$

modified dual constraints are

$$(3.4) \quad \beta \bar{p}_2(t) + \bar{w}(t) = \delta \bar{p}_1(t+1)$$

$$(3.5) \quad \alpha \bar{p}_1(t) + \bar{w}(t) = \delta \bar{p}_2(t+1)$$

$$(3.6) \quad \bar{p}_1(t) \geq \phi_1'(c_1(t))$$

$$(3.7) \quad \bar{p}_2(t) \geq \phi_2'(c_2(t))$$

$$\bar{p}(t), \bar{w}(t) \geq 0 \text{ for } t \geq 0$$

and the complementarity conditions are

$$(3.8) \quad \bar{w}(t)(1 - x_1(t+1) - x_2(t+1)) = 0$$

$$(3.9) \quad c_1(t)(\bar{p}_1(t) - \phi_1'(c_1(t))) = 0$$

$$(3.10) \quad c_2(t)(\bar{p}_2(t) - \phi_2'(c_2(t))) = 0.$$

Note: Here  $(\bar{p}(t), \bar{w}(t))$  are "present" prices, the "true" competitive prices are  $\delta^t(\bar{p}(t), \bar{w}(t))$ .

### 3.2 Instability Property of Boundary OSP

From Chapter 2, we know that there are two boundary OSPs in the two-sector model, namely  $(x^1, c^1)$ ,  $(x^2, c^2)$ . Without loss of generality, we will only show the case when  $(x^1, c^1)$  is an OSP.

We state the main theorem as follows:

#### Theorem 3.1

$(x^1, c^1)$  is unstable if the following conditions are satisfied

$$(3.11) \text{ (a)} \quad \alpha < \sqrt{\alpha\beta} < \delta < \beta < 1$$

$$(3.12) \text{ (b)} \quad \frac{\phi_1'(c_1^1)}{\phi_2'(0)} > \frac{\delta + \beta}{\delta + \alpha}.$$

Note: (1) (b) implies that  $(X^1, C^1)$  is an OSP.

(2) Symmetric condition of (a) for  $(X^2, C^2)$  is  $\beta < \sqrt{\alpha\beta} < \delta < \alpha < 1$ .

(3)  $\delta > \sqrt{\alpha\beta}$  is the  $\delta$ -productivity condition

(4) Condition (a) can be better understood in the  $n$ -sector model, the critical discount factor  $\delta' = \beta$  is actually an eigenvalue of a reduced matrix.

To prove Theorem 3.1, we need the following lemma, which is due to David Gale.

Lemma 3.1

Consider the following recursive relation

$$(3.13) \quad x_{t+1} = -mx_t + b + \varepsilon_t$$

where  $m > 1$  and  $|\varepsilon_t| < \varepsilon$  for  $t \geq 0$ . If

$$(3.14) \quad |x_0 - \frac{b}{m+1}| > \frac{\varepsilon}{m-1} \text{ then } |x_t| \rightarrow \infty.$$

Proof:

Let  $z_t = x_t - \frac{b}{m+1}$ , then (3.13) becomes  $z_{t+1} = -mz_t + \varepsilon_t$  so we have

$$(3.15) \quad |z_{t+1}| \geq m|z_t| - \varepsilon \text{ for all } t, \text{ from (3.14).}$$

We have  $m|z_0| - \varepsilon > |z_0|$ , so there exists  $\tilde{m} > 1$  such that  $m|z_0| - \varepsilon \geq \tilde{m}|z_0|$ . Applying this repeatedly to (3.15) shows that  $|z_t| \geq \tilde{m}^t |z_0|$  so  $|z_t| \rightarrow \infty$ , hence  $|x_t| \rightarrow \infty$ . ■

Forward argument of Theorem 3.1 is given in the following lemma.

Lemma 3.2

Given any optimal program  $(X(t), C(t))_{t=0}^{\infty}$  starting from an initial stock  $X(0)$ , if  $(X(t), C(t))_{t=0}^{\infty}$  converges to  $(X^1, C^1)$  as  $t$  tends to  $\infty$ , then there exists  $T \geq 0$  such that

$$(a) \quad (X(t), C(t)) = (X^1, C^1) \text{ for } t > T$$

$$(b) \quad \text{Let } (\bar{P}(t), \bar{W}(t))_{t=0}^{\infty} \text{ be any competitive prices (modified) for } (X(t), C(t))_{t=0}^{\infty}, \text{ then } (\bar{P}(t), \bar{W}(t)) = (\bar{P}^1, \bar{W}^1) = \left( \phi_1'(c_1^1) \right),$$

$$\frac{\delta + \alpha}{\delta + \beta} \phi_1'(c_1^1), \frac{\det(\delta I - A)}{\delta + \beta} \phi_1'(c_1^1) \text{ for } t > T.$$

Proof:

Theorem 2.5 assures the existence of competitive prices  $(\bar{P}(t), \bar{W}(t))_{t=0}^{\infty}$  for  $(X(t), C(t))_{t=0}^{\infty}$ . Since

$$(3.16) \quad (x_1(t), x_2(t), c_1(t), c_2(t)) \rightarrow (x_1^1, x_2^1, c_1^1, 0) \text{ as } t \rightarrow \infty$$

there exists  $T_1 > 0$  such that  $c_1(t) > 0$  for  $t > T_1$ . (3.9) implies  $\bar{p}_1(t) = \phi_1'(c_1(t))$  for  $t > T_1$ . Continuity of  $\phi_1'$  and (3.16), then gives

$$(3.17) \quad \bar{p}_1(t) \rightarrow \bar{p}_1^1 \text{ as } t \rightarrow \infty.$$

Substracting and rewriting in (3.4) and (3.5), we get

$$(3.18) \quad \begin{aligned} \bar{p}_2(t+1) &= -\frac{\beta}{\delta} \bar{p}_2(t) + \bar{p}_1(t+1) + \frac{\alpha}{\delta} \bar{p}_1(t) \\ &= -\frac{\beta}{\delta} \bar{p}_2(t) + \left(1 + \frac{\alpha}{\delta}\right) \bar{p}_1^1 + \epsilon_t \end{aligned}$$

where

$$\varepsilon_t = \bar{p}_1(t+1) + \frac{\alpha}{\delta} \bar{p}_1(t) - \left(1 + \frac{\alpha}{\delta}\right) \bar{p}_1^{-1}.$$

(3.17) implies  $\varepsilon_t \rightarrow 0$  as  $t \rightarrow \infty$ , hence for any  $\varepsilon > 0$ , there exists  $T_2 \geq T_1$  such that  $|\varepsilon_t| < \varepsilon$  for  $t > T_2$ . By Lemma 3.1 we have

$$\left| \bar{p}_2(t) - \frac{b}{\frac{\beta}{\delta} + 1} \right| \leq \frac{\varepsilon}{\frac{\beta}{\delta} - 1},$$

since otherwise  $|\bar{p}_2(t)| \rightarrow \infty$  as  $t \rightarrow \infty$ , contradicting to Theorem 2.6.

Therefore,

$$(3.19) \quad \bar{p}_2(t) + \frac{b}{\frac{\beta}{\delta} + 1} = \bar{p}_2^{-1} = \frac{\delta + \alpha}{\delta + \beta} \phi_1'(c_1^1) \text{ as } t \rightarrow \infty.$$

Continuity of  $\phi_2'$  and (3.16) gives

$$\phi_2'(c_2(t)) \rightarrow \phi_2'(0) \text{ as } t \rightarrow \infty.$$

By (3.12) and (3.19), there exists  $T_3 \geq T_2$  such that

$$\bar{p}_2(t) > \phi_2'(c_2(t)) \text{ for } t > T_3$$

this implies, by (3.10),  $c_2(t) = 0$  for  $t > T_3$ . Also since  $\bar{w}^1 > 0$ , there exists  $T_4 \geq T_3$  such that

$$x_1(t) = 1 - x_2(t) \text{ for } t > T_4.$$

By using this in (3.2), it gives

$$x_1(t+1) = \frac{1}{\beta} (1 - x_1(t)) \text{ for } t > T_4$$

but this recursion diverges unless  $x_1(t) = \bar{x}_1$ . It follows that  $(X(t), C(t)) = (X^1, C^1)$  for  $t > T_4$  since  $c_1^1 > 0$ , it implies that  $\bar{p}_1(t) = \phi'_1(c_1^1)$  for  $t > T_4$ . From (3.18),  $\varepsilon_t = 0$  for  $t > T_4$ . This implies, by Lemma 3.1,  $\bar{p}_2(t) = \bar{p}_2^1$ , hence we have

$$(\bar{p}_1(t), \bar{p}_2(t), \bar{w}(t)) = (\bar{p}_1^1, \bar{p}_2^1, \bar{w}^1) \text{ for } t > T_4$$

this completes the forward argument. ■

We now complete the proof of Theorem 3.1 with the backward argument.

Proof of Theorem 3.1:

Let  $T_0$  be the smallest  $T$  that satisfy (a) and (b) in Lemma 3.2.

We will prove the theorem in two cases.

Case 1  $T_0 = 0$

We claim that  $c_2(0)$  has to be 0; by definition, it implies that  $(X^1, C^1)$  is unstable.

If  $c_2(0) > 0$ , then

$$(3.20) \quad \bar{p}_2(0) = \phi'_2(c_2(0)) < \phi'_2(0)$$

for any  $\varepsilon > 0$ , we can find a neighborhood  $N(X^1, C^1)$  of  $(X^1, C^1)$  so small that

$$\left| \bar{p}_1(0) - \bar{p}_1^1 \right| = \left| \phi'_1(c_1(0)) - \bar{p}_1^1 \right| < \varepsilon.$$

But by (3.20) and subtracting of (3.4) and (3.5), we have

$$(3.21) \quad \delta \bar{p}_2^1 - \alpha \bar{p}_1(0) = \delta \bar{p}_2^1 - \beta \bar{p}_2(0) > \delta \bar{p}_1^1 - \beta \phi'_2(0) > \delta \bar{p}_1^1 - \beta \bar{p}_2^1$$

be choosing  $\epsilon$  small enough, (3.21) would imply  $\delta p_2^{-1} - \alpha p_1^{-1} > \delta p_1^{-1} - \beta p_2^{-1}$ , which is impossible, since by (3.4) and (3.5), we know  $\delta p_2^{-1} - \alpha p_1^{-1} = \delta p_1^{-1} - \beta p_2^{-1} = \bar{W}^1$  hence  $c_2(0) = 0$ , this proves our claim.

Case 2  $T_0 > 0$

Again, we claim  $c_2(T_0) = 0$ . If not, then from (3.1) - (3.3), we have  $c_1(T_0) < c_1^1$ , this implies, by strict concavity of  $\phi_1$ ,

$$\bar{p}_1(T_0) = \phi_1'(c_1(T_0)) > \phi_1'(c_1^1) = \bar{p}_1^1.$$

By similar argument as in (3.21)

$$\delta p_2^{-1} - \alpha p_1^{-1} \geq \delta p_2^{-1} - \alpha \bar{p}_1(T_0) > \delta p_1^{-1} - \beta p_2^{-1}$$

this is again impossible, hence we have shown  $c_2(T_0) = 0$ . Now by (3.3) and the definition of  $T_0$ , we have

$$(3.22) \quad c_1(T_0) < c_1^1,$$

hence (3.1) and (3.2) imply

$$(3.23) \quad x_1(T_0) < x_1^1 \text{ and } x_2(T_0) = x_2^1.$$

(3.22) implies  $\bar{p}_1(T_0) = \phi_1'(c_1(T_0)) > \phi_1'(c_1^1) = \bar{p}_1^1$ , eliminating  $\bar{W}(T_0)$  from (3.4) and (3.5) gives

$$(3.24) \quad \bar{p}_2(T_0) = \frac{1}{\beta} (\alpha \bar{p}_1(T_0) + (\delta p_1^{-1} - \delta p_2^{-1})) > \frac{1}{\beta} (\alpha \bar{p}_1^1 + \delta p_1^{-1} - \delta p_2^{-1}) = \bar{p}_2^1.$$

By (3.8) and (3.23), we have  $\bar{W}(T_0 - 1) = 0$  hence (3.4), (3.5) and (3.24) imply

$$\bar{p}_1(T_0 - 1) = \frac{\delta}{\alpha} \bar{p}_2(T_0) \geq \frac{\delta}{\alpha} \bar{p}_2^1 = \frac{1}{\alpha} (\alpha \bar{p}_1^1 + \bar{w}^1) > \bar{p}_1^1 = \phi_1'(c_1^1)$$

and

$$\bar{p}_2(T_0 - 1) = \frac{\delta}{\beta} \bar{p}_1(T_0) \geq \frac{\delta}{\beta} \bar{p}_1^1 = \frac{1}{\beta} (\beta \bar{p}_2^1 + \bar{w}^1) > \bar{p}_2^1 > \phi_2'(0) .$$

This would give us

$$c_1(T_0 - 1) < c_1^1 \text{ and } c_2(T_0 - 1) = 0$$

hence by (3.1), (3.2) and (3.23)

$$x_1(T_0 - 1) = c_1(T_0 - 1) + \alpha x_2(T_0) < c_1^1 + \alpha x_2^1 = x_1^1$$

$$x_2(T_0 - 1) = c_2(T_0 - 1) + \beta x_1(T_0) < \beta x_1^1 = x_2^1 .$$

Repeating the above argument  $T_0$  times, we could show  $x_1(0) < x_1^1$  and  $x_2(0) < x_2^1$ , which implies, of course, that  $(x^1, c^1)$  is unstable.

The proof of Theorem 3.1 is now complete. ■

### 3.3 Instability Property of Interior OSP

Let  $(\bar{X}, \bar{C})$  be an interior OSP and let  $(\bar{P}, \bar{W})$  be its modified competitive prices as shown in the proof of Theorem 2.2. As in the boundary case, we will have forward and backward arguments in the analysis of instability. The following statement constitutes the forward argument. We will need a reasonable assumption, which will be given in the process of proving the statement.

(A) There exists a  $\delta' \in (0, 1)$  such that if the discount factor  $\delta$  satisfies  $\sqrt{\alpha\beta} < \delta < \delta'$ , then for any stable optimal program, there exists a  $T > 0$  such that

$$(a) (X(t), C(t)) = (\bar{X}, \bar{C}) \text{ for } t > T$$

(b) let  $(\bar{P}(t), \bar{W}(t))_{t=0}^{\infty}$  be any modified competitive prices for  $(X(t), C(t))_{t=0}^{\infty}$ , then  $(P(t), W(t)) = (\bar{P}, \bar{W})$  for  $t > T$ .

Proof of statement (A):

First, we note that, by (3.9) and (3.10) it is legal to assume that  $(X(t), C(t), P(t), W(t))$  is any optimal program that converges to  $(\bar{X}, \bar{C}, \bar{P}, \bar{W})$ . Since both  $\bar{C}$  and  $\bar{W}$  are positive, it implies that there exists  $T_1 \geq 0$ , such that  $C(t)$  and  $W(t)$  will be positive also, for  $t > T_1$ , hence the following equations should hold for  $t > T_1$

$$(1) \alpha x_2(t+1) + c_1(t) = x_1(t)$$

$$(2) \beta x_1(t+1) + c_2(t) = x_2(t)$$

$$(3) x_1(t+1) + x_2(t+1) = 1$$

$$(4) \delta \phi_1'(c_1(t+1)) + W(t+1) = \beta \phi_2'(c_2(t))$$

$$(5) \delta \phi_2'(c_2(t+1)) + W(t+1) = \alpha \phi_1'(c_1(t)) .$$

(1), (2) and (3) imply

$$(3.25) \quad \begin{aligned} & \beta c_1(t+1) + \alpha c_2(t+1) + c_1(t) + c_2(t) + \alpha \beta \\ & = \alpha x_2(t+1) + \beta x_1(t+1) + c_1(t) + c_2(t) = 1 . \end{aligned}$$

(4) and (5) imply

$$(3.26) \quad \phi_1'(c_1(t+1)) - \phi_2'(c_2(t+1)) = \frac{\beta \phi_2'(c_2(t)) - \alpha \phi_1'(c_1(t))}{\delta} .$$

We can use (3.25) and (3.26) to describe the optimal program. If  $(X(t), C(t))$  is given, then, by (3.25) and (3.26) we can solve for  $C(t+1)$  in terms of  $C(t)$ , also from (1) and (2)  $X(t+1)$  can be

found in terms of  $(X(t), C(t))$ . Hence a stable optimal program for  $t > T$  can be obtained by solving a dynamical system in  $(X(t), C(t))$ .

We first need to check the existence of  $C(t + 1)$  in (3.25) and (3.26).

Jacobian of the system formed by (3.25) and (3.26) is

$$\det \begin{pmatrix} \beta & \alpha \\ \phi_1''(c_1) & -\phi_2''(c_2) \end{pmatrix} = -\beta\phi_2''(c_2) - \alpha\phi_1''(c_1) > 0$$

by inverse function theorem, the existence of  $C(t + 1)$  is guaranteed.

Before we go on, we make the following notation changes. We let  $(X', C') = (X(t + 1), C(t + 1))$  and  $(X, C) = (X(t), C(t))$ . The solutions to (3.25) and (3.26) can then be expressed as

$$\begin{aligned} c_1^1 &\equiv \bar{c}_1(c_1, c_2) \\ c_2^1 &\equiv \bar{c}_2(c_1, c_2) . \end{aligned}$$

By (3), (1) and (2) can be rewritten as

$$(1') \quad \alpha(1 - x_1') + c_1 = x_1$$

$$(2') \quad \beta x_1' + c_2 = 1 - x_1$$

eliminating  $x_1'$  from (1') and (2') gives

$$(3.27) \quad c_2 = \frac{\beta - \alpha}{\alpha} x_1 - \frac{\beta}{\alpha} c_1 + 1 - \beta \equiv t(x_1, c_1)$$

therefore, the dynamical system we have described above can be reduced to the following 2-dimensional system

$$\begin{cases} x'_1 = \frac{1}{\beta} (1 - x_1 - t(x_1, c_1)) \\ c'_1 = \bar{c}_1(c_1, t(x_1, c_1)) \end{cases} .$$

Now we compute the Jacobian of this reduced system

$$\begin{aligned} Q &= \begin{pmatrix} \frac{\partial x'_1}{\partial x_1} & \frac{\partial x'_1}{\partial c_1} \\ \frac{\partial c'_1}{\partial x_1} & \frac{\partial c'_1}{\partial c_1} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{\alpha} & \frac{1}{\alpha} \\ \frac{\beta - \alpha}{\alpha} \frac{\partial \bar{c}_1}{\partial c_2} & \frac{\partial \bar{c}_1}{\partial c_1} - \frac{\beta}{\alpha} \frac{\partial \bar{c}_1}{\partial c_2} \end{pmatrix}. \end{aligned}$$

Applying chain rule on (3.25) and (3.26) we can compute the following equations

$$\frac{\partial \bar{c}_1}{\partial c_1} = -\frac{1}{K} \left( \phi_2'' + \frac{\alpha^2}{\delta} \phi_1'' \right)$$

$$\frac{\partial \bar{c}_2}{\partial c_1} = \frac{1}{K} \left( -1 + \frac{\alpha\beta}{\delta} \right) \phi_1''$$

$$\frac{\partial \bar{c}_1}{\partial c_2} = \frac{1}{K} \left( -1 + \frac{\alpha\beta}{\delta} \phi_2'' \right)$$

$$\frac{\partial \bar{c}_2}{\partial c_2} = -\frac{1}{K} \left( \phi_1'' + \frac{\beta^2}{\delta} \right) \phi_2''$$

$$\text{here } K = \alpha\phi_1'' + \beta\phi_2'' .$$

Hence we are able to compute

$$Q = \begin{pmatrix} -\frac{1}{\alpha} & \frac{1}{\alpha} \\ \frac{\beta - \alpha}{\delta \alpha K} (-\delta + \alpha \beta) \phi_2''' & \frac{-1}{\delta \alpha K} (\alpha^3 \phi_1''' + (\beta \alpha - \delta \beta + \alpha \beta^2) \phi_2''') \end{pmatrix}.$$

The characteristic equation of  $Q$  is

$$(3.28) \quad \det(Q - \lambda I) = \lambda^2 + \frac{(\phi_1''' + \phi_2''') + \frac{1}{\delta} (\alpha^2 \phi_1''' + \beta^2 \phi_2''')}{\alpha \phi_1''' + \beta \phi_2'''} \lambda + \frac{1}{\delta} \\ \equiv f_\delta(\lambda).$$

Let the two roots of  $f_\delta(\lambda)$  be  $\lambda_1(\delta)$ ,  $\lambda_2(\delta)$ , we choose the indices of  $\lambda_i(\delta)$  such that  $|\lambda_1(\delta)| \geq |\lambda_2(\delta)|$ . It is easy to see that  $\lambda_i(\delta)$  is a continuous function of  $\delta$  for  $i = 1, 2$ .

To continue our argument, we have to make a reasonable assumption.

Assumption:

$$|\lambda_i(1)| \neq 1 \text{ for } i = 1, 2.$$

Remark:

Without the assumption, turnpike property might not hold at  $\delta = 1$  (undiscounted case).

From (3.28), we have  $\lambda_1(1) \cdot \lambda_2(1) = 1$ , this, with the assumption above, gives us

$$|\lambda_1(1)| > 1 > |\lambda_2(1)|$$

by (3.28) again, we have

$$(3.29) \quad \lambda_1(\delta) \cdot \lambda_2(\delta) = \frac{1}{\delta} > 1.$$

In order for the existence of  $\delta'$  that satisfies (A),  $\delta'$  has to

satisfy the following conditions

- (a) for  $\delta' < \delta \leq 1$   $|\lambda_1(\delta)| > 1 > |\lambda_2(\delta)|$
- (b) for  $\delta = \delta'$   $|\lambda_1(\delta)| > |\lambda_2(\delta)| = 1$
- (c) for  $\delta < \delta'$   $|\lambda_1(\delta)| \geq |\lambda_2(\delta)| > 1$ .

(3.29) and (b) imply that  $\lambda_1(\delta')$  and  $\lambda_2(\delta')$  cannot be complex conjugates hence  $\lambda_2(\delta')$  must be  $+1$  or  $-1$ , this gives us way to compute  $\delta'$ .

Plug  $\pm 1$  in (3.28) and solve for  $\delta$ , we get  $\delta_+$ ,  $\delta_-$  as follows

$$\delta_+ = -\frac{\alpha(1+\alpha)\phi_1'' + \beta(1+\beta)\phi_2''}{(1+\alpha)\phi_1'' + (1+\beta)\phi_2''}$$

$$\delta_- = \frac{\alpha(\alpha-1)\phi_1'' + \beta(\beta-1)\phi_2''}{(\alpha-1)\phi_1'' + (\beta-1)\phi_2''}.$$

It is easy to check that

$$0 < \delta_- < 1 \text{ and } -1 < \delta_+ < 0$$

hence  $\delta' = \delta_-$  and (A) is proved. ■

Backward argument for the interior case is similar to that of the boundary case. First, we let  $T_0$  be the smallest  $T$  such that statement (A) is true. Again, we divide it into two cases.

Case 1  $T_0 = 0$

By (4) and (5)

$$\phi_1'(c_1(0)) = \frac{1}{\alpha} (\delta \bar{p}_2 - w(0))$$

$$\phi_2'(c_2(0)) = \frac{1}{\beta} (\delta \bar{p}_1 - w(0)).$$

Since  $\phi_{11} < 0$  and  $\phi_{22} < 0$ , by inverse function theorem, there

exists two differentiable functions  $g$  and  $h$  such that  $c_1(0) = g(W(0))$ ,  $c_2(0) = h(W(0))$  where  $W(0)$  is a parameter, hence by (1) and (2),  $X(0)$  can only belong to a 1-dimensional manifold.

Case 2  $T_0 > 0$

By (4) and (5)

$$\begin{aligned}\phi'_1(c_1(T_0)) &= \frac{1}{\alpha} (\delta \bar{p}_2 - W(T_0)) \\ \phi'_2(c_2(T_0)) &= \frac{1}{\beta} (\delta \bar{p}_1 - W(T_0)) .\end{aligned}$$

We claim  $W(T_0) < \bar{W}$ . If not, then we have

$$\begin{aligned}\phi'_1(c_1(T_0)) &< \bar{p}_1 = \phi'_1(\bar{c}_1) \\ \phi'_2(c_2(T_0)) &< \bar{p}_2 = \phi'_2(\bar{c}_2) .\end{aligned}$$

This implies  $c_1(T_0) > \bar{c}_1$  and  $c_2(T_0) > \bar{c}_2$ . Hence, by (1) and (2),  $x_1(T_0) + x_2(T_0) > 1$  is impossible. Therefore,

$$\phi'_1(c_1(T_0)) = \frac{1}{\alpha} (\delta \bar{p}_2 - W(T_0)) > \frac{1}{\alpha} (\delta \bar{p}_2 - \bar{W}) = \bar{p}_1 = \phi'_1(\bar{c}_1)$$

$$\phi'_2(c_2(T_0)) = \frac{1}{\beta} (\delta \bar{p}_1 - W(T_0)) > \frac{1}{\beta} (\delta \bar{p}_1 - \bar{W}) = \bar{p}_2 = \phi'_2(\bar{c}_2)$$

hence  $c_1(T_0) < \bar{c}_1$  and  $c_2(T_0) < \bar{c}_2$ , (1) and (2) imply

$$(3.30) \quad x_1(T_0) < \bar{x}_1 \text{ and } x_2(T_0) < \bar{x}_2 .$$

This gives  $x_1(T_0) + x_2(T_0) < 1$ . By (3.8),  $W(T_0 - 1) = 0$ . (4) and (5) again gives

$$\phi_1(c_1(T_0 - 1)) = \frac{1}{\alpha} \delta p_2(T_0) > \frac{1}{\alpha} (\delta \bar{p}_2 - \bar{w}) = \bar{p}_1$$

$$\phi_2(c_2(T_0 - 1)) = \frac{1}{\beta} \delta p_2(T_0) > \frac{1}{\beta} (\delta \bar{p}_1 - \bar{w}) = \bar{p}_2 .$$

Hence  $c_1(T_0 - 1) < \bar{c}_1$  and  $c_2(T_0 - 1) < \bar{c}_2$ , from (3.30), (1) and (2) we have

$$x_1(T_0 - 1) < \bar{x}_1 \text{ and } x_2(T_0 - 1) < \bar{x}_2 .$$

Repeating this argument  $T_0$  times, we could show

$$x_1(0) < \bar{x}_1 \text{ and } x_2(0) < \bar{x}_2 .$$

Backward argument is now complete.

### 3.4 An Example

$$\text{Let } A = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} = \begin{pmatrix} 0 & .4 \\ .8 & 0 \end{pmatrix}$$

$$u(c_1, c_2) = (-4c_1^2 + 15c_1) + (-15c_2^2 + 15c_2) = \phi_1(c_1) + \phi_2(c_2) .$$

The two boundary stationary programs given as before are

$$(x^1, c^1) = \left( \frac{5}{9}, \frac{4}{9}, \frac{3.4}{9}, 0 \right)$$

$$(x^2, c^2) = \left( \frac{2}{7}, \frac{5}{7}, 0, \frac{3.4}{7} \right) .$$

Differentiating  $u(c_1, c_2)$ , we get

$$\phi'_1(c_1) = -8c_1 + 15$$

$$\phi'_2(c_2) = -30c_2 + 15 .$$

By theorem 2.2, if  $(X, C)$  is an interior OSP, then there exists a  $\lambda \in (0,1)$  such that

$$(X, C) = \lambda(X^1, C^1) + (1 - \lambda)(X^2, C^2)$$

and

$$\frac{\phi'_1(c_1)}{\phi'_2(c_2)} = \frac{-8c_1 + 15}{-30c_2 + 15} = \frac{\delta + .8}{\delta + .4} .$$

Solving the above two equations for  $\lambda$  in terms of  $\delta$ , we get  
 $\lambda = \frac{14.57\delta + 5.66}{17.59\delta + 12.87}$  for any  $\delta \in (0,1)$ . Since  $\lambda \in (0,1)$ , hence, there exists an interior OSP for  $\delta > \sqrt{\alpha\beta} = \bar{\delta} = .566$  ( $\delta$ -productivity).

Now we compute  $\delta'$  as follows

$$\delta' = \frac{\alpha(\alpha - 1)\phi''_1 + \beta(\beta - 1)\phi''_2}{(\alpha - 1)\phi''_1 + (\beta - 1)\phi''_2} = .622 .$$

Hence for  $\delta \in (0.566, 0.622)$ , the interior OSP will be unstable.

### 3.5 General $z$ -sector Model

We have seen in the last two sections the instability arguments for both types of OSPs in the special two-sector model. Most of the arguments are still valid when we generalize it to any two-sector model. The argument still consists of the same two stages.

We shall only briefly describe what will happen in each case. For the boundary OSP, condition (a) and (b) of Theorem 3.1 could be replaced as follows

$$(a') \quad \bar{\delta} < \delta < |a_{21} - a_{22}| < 1$$

$$(b') \frac{\delta + a_{12} - a_{11}}{\delta + a_{21} - a_{22}} \phi_1(c_1^1) > \phi_2(0)$$

where  $\bar{\delta}$  comes from  $\delta$ -productivity assumption.

For the interior case, again, we could set up a two-dimensional dynamical system, a discount factor  $\delta'$  could then be obtained similar to the procedure in the special case. An appropriate backward argument is needed to show that a stable optimal program can only start from a set of measure 0. We will not go into the details here.

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